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COMMENT

Off-shell electromagnetic duality invariance

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Abstract. We review the theorem that the Maxwell action, and not just the field equation, is invariant under duality transformations.

The concept of duality transformations originates in the manifest formal invariance of the source-free Maxwell equations under arbitrary linear transformations of the field strength into its dual, in particular under the rotations

$$F'_{\mu\nu} = F_{\mu\nu} \cos\theta + {}^*F_{\mu\nu} \sin\theta, \qquad {}^*F^{\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{1}$$

This formal transformation was shown some time ago (Deser and Teitelboim 1976) to be implementable by a conserved generator functional of the dynamical variables; furthermore it is also an invariance of the Maxwell action itself. Nevertheless, there still seems to be some confusion about its off-shell validity. The purpose of this note is to clarify precisely in what sense this is the case; we emphasise that all the results are already present in the cited reference, where further details may be found.

The unconstrained first-order Maxwell action is

$$I[\boldsymbol{E}^{\mathrm{T}}, \boldsymbol{A}^{\mathrm{T}}] = -\int d^4x \{\boldsymbol{E}^{\mathrm{T}} \cdot \dot{\boldsymbol{A}}^{\mathrm{T}} + \tfrac{1}{2}[\boldsymbol{E}^{\mathrm{T}2} + (\boldsymbol{\nabla}\times\boldsymbol{A}^{\mathrm{T}})^2]\} \tag{2}$$

where the superscript emphasises the transverse ($\boldsymbol{\nabla}\cdot\boldsymbol{V}^{\mathrm{T}} \equiv 0$) nature of the independent variables ($\boldsymbol{E}^{\mathrm{T}}, \boldsymbol{A}^{\mathrm{T}}$). We have eliminated the Gauss constraint $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$ (together with its associated Lagrange multiplier A_0) for convenience only. The constraint is not a true field equation, since the dynamics is always describable by the reduced set ($\boldsymbol{E}^{\mathrm{T}}, \boldsymbol{A}^{\mathrm{T}}$). We have not imposed any gauge conditions on $\boldsymbol{A}$, but only exploited the orthogonality of transverse and longitudinal vectors, so that $\int \boldsymbol{E}^{\mathrm{T}}\cdot\boldsymbol{\nabla}A^{\mathrm{L}}\, d^3x = 0$ (with obvious boundary conditions), and also used $\boldsymbol{\nabla}\times\boldsymbol{\nabla}A^{\mathrm{L}} \equiv 0$. In this space of transverse vectors, the duality operation is defined by (dropping the T superscript)

$$\begin{aligned} \boldsymbol{E}' &= \boldsymbol{E}\cos\theta + \boldsymbol{\nabla}\times\boldsymbol{A}\sin\theta, \\ \boldsymbol{A}' &= \boldsymbol{A}\cos\theta + \boldsymbol{\nabla}^{-2}\boldsymbol{\nabla}\times\boldsymbol{E}\sin\theta \Leftrightarrow \boldsymbol{B}' = \boldsymbol{B}\cos\theta - \boldsymbol{E}\sin\theta, \end{aligned} \tag{3}$$

where $\boldsymbol{B} \equiv \boldsymbol{\nabla}\times\boldsymbol{A}$. It is implemented by the conserved generator (obtained from the Noether current)

$$G = \tfrac{1}{2}\int d^3x\, [\boldsymbol{E}\cdot\boldsymbol{\nabla}^{-2}\boldsymbol{\nabla}\times\boldsymbol{E} - \boldsymbol{A}\cdot\boldsymbol{\nabla}\times\boldsymbol{A}]. \tag{4}$$

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It is a matter of simple algebra to verify that the following invariances hold under (3):

$$H[\boldsymbol{E}, \boldsymbol{A}] \equiv \tfrac{1}{2} \int \mathrm{d}^3x\, [\boldsymbol{E}^2 + (\boldsymbol{\nabla} \times \boldsymbol{A})^2] = H[\boldsymbol{E}', \boldsymbol{A}'], \tag{5a}$$

$$[\boldsymbol{E}(\boldsymbol{r}), \boldsymbol{A}(\boldsymbol{0})] = [\boldsymbol{E}'(\boldsymbol{r}), \boldsymbol{A}'(\boldsymbol{0})] \tag{5b}$$

and most important

$$I[\boldsymbol{E}, \boldsymbol{A}] = I[\boldsymbol{E}', \boldsymbol{A}']. \tag{5c}$$

The last equation follows from the fact that the $\boldsymbol{E} \cdot \boldsymbol{A}$ term in (2) is separately invariant, as might be expected from (5*b*) since $\boldsymbol{E} \cdot \delta\boldsymbol{A}$ generates canonical transformations. This establishes off-shell invariance of the theory in first-order form.

Although we have worked in flat space for simplicity, everything carries through to curved space, so that the Einstein–Maxwell system is also invariant (with untransformed metric) in terms of an appropriately defined $\mathscr{E}^i \equiv \sqrt{-g}F^{0i}$.

In conventional second-order form (where $\boldsymbol{E} \equiv -\dot{\boldsymbol{A}}$ is an identity rather than a field equation as in first order) the action reads

$$I[\boldsymbol{A}] = \tfrac{1}{2} \int \mathrm{d}^4x\, [\dot{\boldsymbol{A}}^2 - (\boldsymbol{\nabla} \times \boldsymbol{A})^2]. \tag{6}$$

It is invariant under the transformation

$$\boldsymbol{B}' = \boldsymbol{B} \cos\theta - \boldsymbol{E} \sin\theta, \tag{7a}$$

whose time derivative is

$$\boldsymbol{E}' = \boldsymbol{E} \cos\theta + \boldsymbol{\nabla}^{-2} \boldsymbol{\nabla} \times \ddot{\boldsymbol{A}} \sin\theta. \tag{7b}$$

However, this is only a duality transformation (in the strict sense that the last term in (7*b*) is $\boldsymbol{B} \sin\theta$) on-shell, i.e. when the field equation $\ddot{\boldsymbol{A}} = \nabla^2 \boldsymbol{A}$ is used. Thus both formulations are off-shell invariant under 'duality' rotations, but only in first-order form is the transformation precisely that of (1).

Some final comments:

(a) The non-compact (hyperbolic) version of (1) is *not* an invariance of the action, despite its $\int (\boldsymbol{E}^2 - \boldsymbol{B}^2)$ form. For then (5*a*) becomes invalid in first-order description, while the transformation itself cannot even be implemented in second order.

(b) Duality transformations are not implementable in non-Abelian theories, basically because the latter involve minimal (self-) coupling. Nevertheless there is an effective partial invariance there which can be exploited to explain the absence of certain one-loop counterterms in the Einstein–Yang–Mills system (Deser 1981).

(c) We mention that the linearised (but not the full) Einstein action also enjoys duality properties similar to those of the Maxwell case.

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References

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